The Four Bugs Problem

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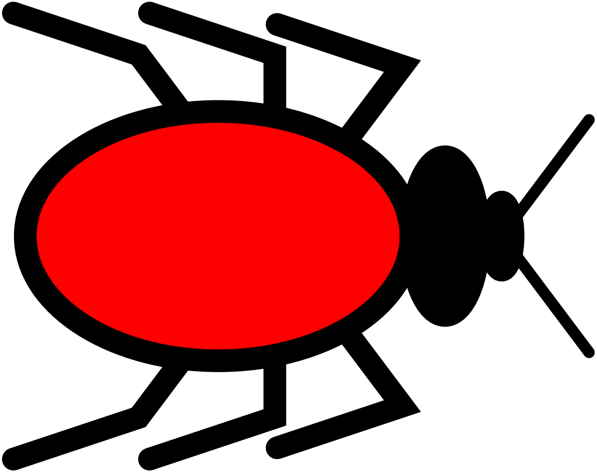
March 3, 2010

1 The four bugs problem



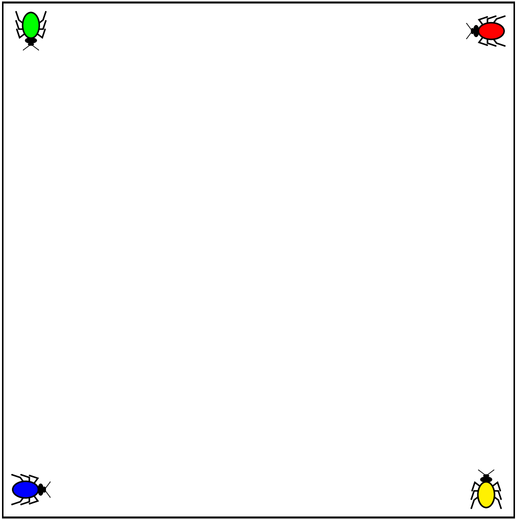
The following is an exercise in *Differential Equations* by Lester R. Ford [1]:

> Four flies sit at the corners of a card table, facing inward. They start simultaneously walking at the same rate, each directing its motion steadily toward the fly on its right. Find the path of each. Can you find without calculus the distance traveled?

The problem is also given in Eli Maor's *e: The Story of a Number* [2] in terms of bugs. Bugs are the featured insects here.

The objective is to solve the four bugs problem.

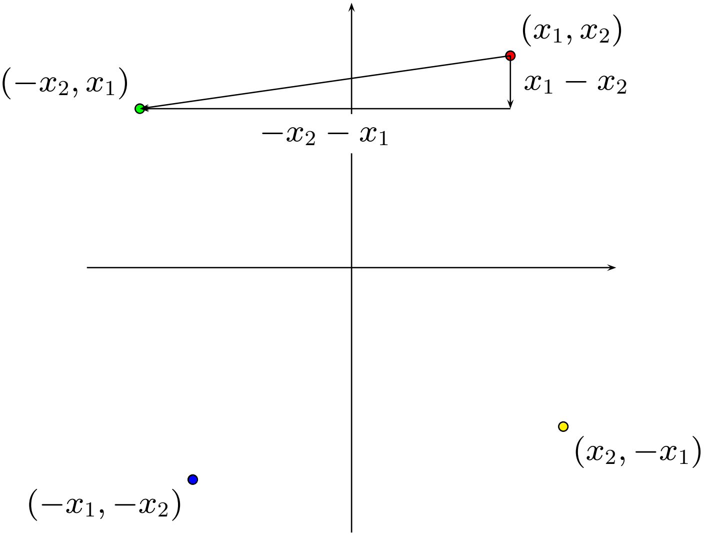
The Starting Positions



The differential equations

The first task is to translate the verbal statement of the problem into a system of differential equations. The discussion focuses on the red bug.

Set the origin of the coordinate system at the center of the table. Let $(x_1(t), x_2(t))$ be coordinates of the red bug at time t. The coordinates of the other bugs are simply expressed as functions of $x_1(t)$ and $x_2(t)$.


(x_1, x_2)
$(-x_2, x_1)$
$x_1 - x_2$
$-x_2 - x_1$
$(x_2, -x_1)$
$(-x_1, -x_2)$

The equations are

$$\dot{x}_1(t) = \theta(-x_2(t) - x_1(t))$$
$$\dot{x}_2(t) = \theta(x_1(t) - x_2(t)).$$

A rearrangement gives

$$\begin{aligned} \dot{x}_1(t) &= -\theta x_1(t) - \theta x_2(t) \\ \dot{x}_2(t) &= \theta x_1(t) - \theta x_2(t). \end{aligned} \tag{1}$$

The discriminant $d = -\theta^2$ is negative.

The solution

The second and final task is the application of the formulas given earlier giving

$$x_1(t) = (\cos(\theta t)x_1(0) - \sin(\theta t)x_2(0))\exp(-\theta t)$$
$$x_2(t) = (\sin(\theta t)x_1(0) + \cos(\theta t)x_2(0))\exp(-\theta t).$$

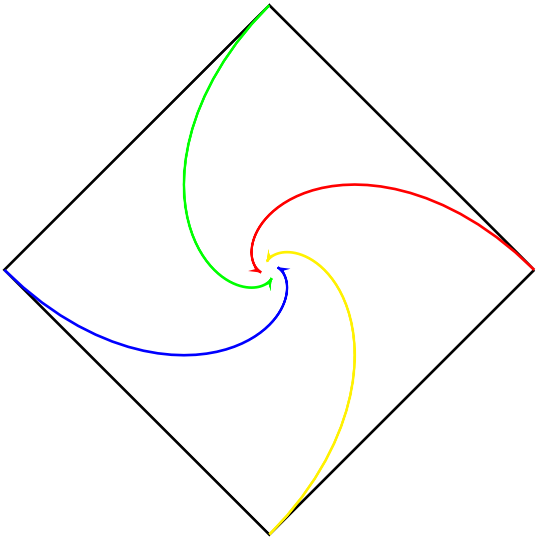
If we rotate the table 45 degrees clockwise, the red bug starts on the horizontal axis, and the formula for his path is simplified. With $x_1(0) = D$, one-half the diagonal measurement of the table, and $x_2(0) = 0$ the formula is

$$\begin{aligned} x_1(t) &= D\cos(\theta t)\exp(-\theta t) \\ x_2(t) &= D\sin(\theta t)\exp(-\theta t). \end{aligned} \tag{2}$$

Logarithmic Spiral

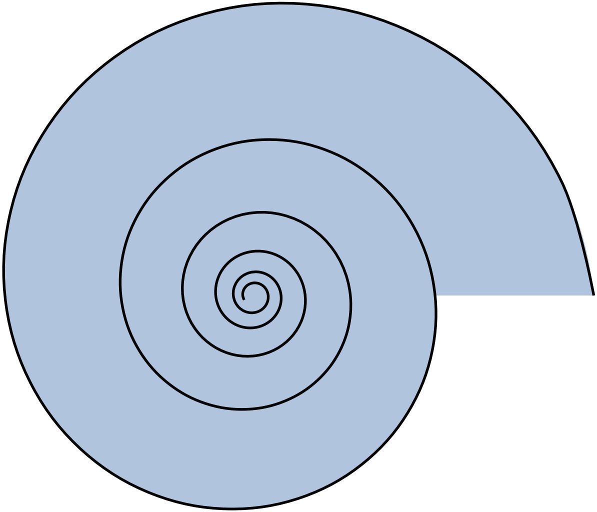
The formulas (2) are the answer to the first part of the problem. What about the other part? The formulas are an example of a *Logarithmic Spiral*. Maor [2] gives a delightful account of the properties of the logarithm spiral and its occurrence in art and nature.

This section concludes with some graphs and pictures.



The bugs paths.

A spiral aloe in the San Francisco Arboretum.



A single spiral.

References

[1] Lester R. Ford. *Differential equations*. McGraw-Hill Book Company, Inc., New York, second edition, 1955. First edition published in 1933.

[2] Eli Maor. e*: The Story of a Number*. Princeton University Press, Princeton, New Jersey, 1994.