Solving Systems of Linear Differential Equations with Real Eigenvalues

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February 18, 2013
This section shows how to find solutions to linear systems of differential equations when the eigenvalues of the system matrix are all real. Finding solutions when there are complex eigenvalues is considerably more difficult. But in an instructional setting, most of the concepts can be conveyed within the real eigenvalue context.
A Quasi-triangular Matrix

A real matrix that is lower block-triangular with each diagonal block being either a $1 \times 1$ matrix or $2 \times 2$ matrix with complex eigenvalues is said to be a lower quasi-triangular matrix. A schematic representation is

$$\begin{bmatrix}
A_{11} & 0 & 0 & \cdots & 0 \\
A_{21} & A_{22} & 0 & \cdots & 0 \\
A_{31} & A_{32} & A_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn}
\end{bmatrix}$$
Fact 1.1. Let $A \in \mathbb{R}^{n \times n}$. Then there exists an orthogonal matrix $R$ such that $R^t A R$ is quasi-triangular. Moreover, $R$ may be chosen so that any $2 \times 2$ diagonal block of $R^t A R$ has only complex eigenvalues (which must therefore be conjugates).

This decomposition is called the \textit{Real Schur form}.

When all eigenvalues of $A$ are real (as assumed in this presentation), Fact 1.1 implies $R^t A R$ is triangular.
A Transformation to Lower Triangular Form

The system to be solved is

\[ \dot{x}(t) = Ax(t) \]  

with \( x(0) \) given. If \( A \) in (1) is not lower triangular, by Fact 1.1 one may compute an orthogonal matrix \( R \) such that \( R^tAR \) is lower triangular. The algorithm for finding \( R \) is far too complicated to present in this class. The algorithm is an intermediate step in solving non-symmetric eigen systems. Refer to the collection of papers in Wilkinson and Reinsch [2].
Recasting to a Triangular System

The system (1) is recast

\[
\dot{x}(t) = Ax(t) \\
(R^t\dot{x}(t)) = (R^tAR)(R^tx(t)) \\
\dot{u}(t) = A'u(t)
\]

The algorithm presented in proof of Fact 1.2 that follows, shows how to solve the triangular system. The solution is then back transformed \(x(t) = Ru(t)\).
**Fact 1.2.** Assume a lower triangular matrix $A$ has real elements. The eigenvalues of $A$ are its diagonal elements, so let $\lambda_i = a_{ii}, i = 1, \cdots, n$. Construct a vector $e(t)$ such that $e_i(t) = t^{k_i} \exp(\lambda_i t)$ where $k_i$ is the number of occurrences of $\lambda_j = \lambda_i, j = 1, \cdots, i - 1$. Consider the equations

$$\dot{x}(t) = Ax(t)$$

(2)

with $x(0)$ given. The solution of is of the form

$$x(t) = Be(t)$$

(3)

where $B$ is lower triangular matrix.
Construction of $e(t)$ Illustrated

To illustrate the construction of $e(t)$, suppose the diagonal elements of $A$, the eigenvalues of $A$, are 0, -2, -3, -2, -3. Then

$$e(t) = \begin{bmatrix} 1 \\ \exp(-2t) \\ \exp(-3t) \\ t \exp(-2t) \\ t \exp(-3t) \end{bmatrix}$$

Consider $i = 4$; $\lambda_i = -2$ and -2 has occurred one time previously, hence, $k_i = 1$. 
The Proof

The proof of Fact 1.2 is constructive in that it gives the recipe for computing $B$. The proof is by induction. Begin by initializing all elements of $B$ to zero. For $i = 1$ let $b_{1,1} = x_1(0)$. Now, with $i > 1$ assume $x_l(t) = \sum_{j=1}^{l} b_{l,j} e_j(t)$ for $l = 1, \ldots, i - 1$. The $i$th equation is

$$\dot{x}_i(t) - \lambda_i x_i(t) = \sum_{j=1}^{i-1} a_{i,j} x_j(t)$$

$$= \sum_{j=1}^{i-1} c_j e_j(t)$$

where $c_j = \sum_{l=1}^{i-1} a_{i,l} b_{l,j}$. 
Multiply Equation (4) by \( \exp(-\lambda_i t) \):

\[
\exp(-\lambda_i t)(\dot{x}_i(t) - \lambda_i x_i(t)) = \sum_{j=1}^{i-1} c_j \exp(-\lambda_i t)e_j(t) \tag{5}
\]

Substitute the definition of \( e_j(t) \) into Equation (5) and integrate both sides:

\[
\exp(-\lambda_i t)x_i(t) = \sum_{j=1}^{i-1} c_j \int \exp((\lambda_j - \lambda_i) t) t^{k_j} \, dt + d \tag{6}
\]

where \( d \) is a constant of integration.
Multiply Equation (6) by \(\exp(\lambda_i t)\):

\[
\chi_i(t) = \sum_{j=1}^{i-1} c_j \exp(\lambda_i t) \int \exp((\lambda_j - \lambda_i) t) t^{kj} \, dt + d \exp(\lambda_i t).
\]

(7)

The integral in each term in equation (7) is to be evaluated. They take a different form depending on the equality or not of \(\lambda_j\) and \(\lambda_i\).
Case where $\lambda_j = \lambda_i$

When $\lambda_j = \lambda_i$, the $j$th term in Equation (7) evaluates to

$$\frac{c_j}{k_j + 1} t^{k_j+1} \exp(\lambda_i t).$$

Let $l$ be the index of the next term for which $l > j$ and $\lambda_l = \lambda_j$. Add $\frac{c_j}{k_l}$ to $b_{i,l}$. 
Case where $\lambda_j \neq \lambda_i$

When $\lambda_j \neq \lambda_i$, the $j$th term in Equation (7) evaluates to

$$c_j \sum_{h=0}^{k_j} \frac{k_j!}{(k_j - h)!(\lambda_i - \lambda_j)^{h+1}} t^{k_j-h} \exp(\lambda_j t).$$

(8)

With $h = 0$ add $\frac{c_j}{\lambda_i - \lambda_j}$ to $b_{i,j}$. If $k_j > 0$ set $l = j$. For each $h > 0$, find the next smaller value of $l$ such that $\lambda_l = \lambda_j$, and add $\frac{c_j k_j!}{(k_j-h)!(\lambda_i - \lambda_j)^{h+1}}$ to $b_{i,l}$. 
Finding the value of $d$

Let $l$ represent the index of the first term where $\lambda_l = \lambda_i$. Let

$$c = \sum_{j=1}^{i} (k_j = 0) b_{i,j}$$

and add $x(0) - c$ to $b_{i,l}$.

Proof of Fact 1.2 is complete.
A Numerical Illustration

The transformation to triangular form is illustrated numerically here. Consider the kinetic diagram

GI tract $\xrightarrow{\theta_1}$ Plasma $\xrightarrow{\theta_2}$ Other
Plasma $\xrightarrow{\theta_3}$ Other $\xrightarrow{\theta_4}$ GI tract
The System Matrix

With $\theta_1 = 4$, $\theta_2 = 2$, $\theta_3 = 3$, and $\theta_4 = 1$, the system matrix is

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 4 & -3 & 3 \\ 0 & 2 & -3 \end{bmatrix}$$

The initial conditions are $x(0) = (100, 0, 0)^t$. 

Back
The Orthogonal Matrix

The orthogonal matrix that will transform $A$ to Schur form is

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6325 & -0.7746 \\ 0 & 0.7746 & 0.6325 \end{bmatrix}$$

The Schur matrix is

$$A' = R^tAR$$

$$= \begin{bmatrix} -4 & 0 & 0 \\ 2.5298 & -0.5505 & 0 \\ -3.0984 & -1 & -5.4495 \end{bmatrix}$$

and $u(0) = R^t x(0)$. 
Solution of the Transformed System

Apply the algorithm presented above. The solution of the transformed system is

\[ u(t) = B e(t) \]

\[
= \begin{bmatrix} 100 & 0 & 0 \\ -73.3390 & 73.3390 & 0 \\ -163.1606 & -14.9703 & 178.1309 \end{bmatrix} \begin{bmatrix} \exp(-4t) \\ \exp(-0.5505t) \\ \exp(-5.4495t) \end{bmatrix}
\]
The solution of the original system is
\[ x(t) = R u(t) \]
\[
= \begin{bmatrix}
100 & 0 & 0 \\
80 & 57.9796 & -137.9796 \\
-160 & 47.3401 & 112.6599
\end{bmatrix}
\begin{bmatrix}
\exp(-4t) \\
\exp(-0.5505t) \\
\exp(-5.4495t)
\end{bmatrix}
\]
**Exercise 1.1.** Assume the same model and initial conditions as given in the numerical example above, but with \( \theta_1 = 2, \theta_2 = 1, \theta_3 = 3, \) and \( \theta_4 = 4 \). Find the solution for \( x(t) \) expressed as \( Be(t) \).